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INTEGRATION OF DYNAMIC EQUATIONS WITH CONSTRAINT MULTIPLIERS

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A. S. SUMBATOV
(Moscow)
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By generalizing the unfinished investigation of Jacobi [1], Suslov showed in [2] and [3] that, knowing the complete integral W of a first order partial differential equation we can construct a set of first integrals of the equations of motion of a mechanical system with constraint multipliers. Suslov confined himself to the case when the constraints imposed on the system are specified in a finite form. For such systems the set of the first integrals mentioned above defines the general solution of the equations of motion, thus providing a method of integrating these equations.

A problem of extending this method to cover nonholonomic systems engaged the efforts of the authors of [4 - 10], [9 - 12], [13, 14]. The author of [6] established that in the case of nonholonomic constraints the integral W must satisfy an additional system of equations, therefore the set of first integrals indicated by Suslov, generally speaking, defines a particular solution of the equations of motion. The authors of [12, 13] (see also the dissertation of E. Kh. Naziev "Certain Problems of Analytical Dynamics", MGU, 1969) obtained the necessary and sufficient conditions which must be imposed on the equations of constraints and on the integral W, in an explicit form. When these conditions hold, the Suslov method can be used to obtain a particular solution of equations with multipliers of nonholonomic constraints. The conditions however are all different. In Sect. 1 of the present paper we show that the most general of these conditions [12] are not necessary and we obtain the necessary conditions. In Sect. 2 we prove the sufficiency of our conditions and show that the sufficiency of [10, 12] conditions follows as a particular case.

Compatibility of the equations determining the integral W was studied in [10, 12] by the usual method of constructing all possible Poisson brackets. In Sects. 3 and 4 we show that the integral W needs not, in fact, satisfy the equations obtained by making the Poisson brackets all equal to zero. The case of a homogeneous sphere rolling on a plane without slipping is used as an example.

1. Let $q_1, \ldots, q_n$ be the generalized coordinates of a mechanical system the constraints of which are given by the equations

$$\sum_{j=1}^{n} A_{ij}(t, q_1, \ldots, q_n)\, q_j^{\cdot} + A_{i0}(t, q_1, \ldots, q_n) = 0 \qquad (i = 1, \ldots, k < n) \qquad (1.1)$$

Equations of motion can be written in canonical variables q and p ($p_j = \partial L / \partial q_j^{\cdot}$), and L is the Lagrangian function of the system

$$H = \sum_{j=1}^{n} p_j q_j^{\cdot} - L$$

$$q_j^{\cdot} = \frac{\partial H}{\partial p_j}, \qquad p_j^{\cdot} = -\frac{\partial H}{\partial q_j} + \sum_{i=1}^{k} \lambda_i A_{ij} \qquad (j = 1, \ldots, n) \tag{1.2}$$

where λ_i are the multipliers of constraints (1.1). For the constraints given in a finite form, i.e. for

$$A_{ij} = \frac{\partial F_i}{\partial q_j}, \qquad A_{i0} = \frac{\partial F_i}{\partial t} \qquad (j = 1, \ldots, n; \quad i = 1, \ldots, k)$$

Suslov showed, after calculating the total variation of the action function $W(t, t_0, q_1, \ldots, q_n, q_1^0, \ldots, q_n^0) = \int_{t_0}^{t} L dt$ with respect to the coordinates q and initial conditions q^0 compatible with the constraints, that equations (1.2) admit the following first integrals

$$p_j = \frac{\partial W}{\partial q_j} + \sum_{i=1}^{k} M_i A_{ij} \tag{1.3}$$

$$\partial W / \partial a_j = b_j \qquad (j = 1, \ldots, n) \tag{1.4}$$

Here M_i denote certain functions which can be determined by inserting the expressions (1.3) for the impulses into the constraint equations and solving the resulting system of algebraic equations for M_i. We note that the corresponding determinant

$$\det \left\| \sum_{j,h=1}^{n} A_{ij} \frac{\partial^2 H}{\partial p_j \partial p_h} A_{lh} \right\|$$

is always positive provided that the rank $\| A_{ij} \| = k$ and $W(t, q_1, \ldots, q_n, a_1, \ldots, a_n)$ is the complete integral of the equation

$$\frac{\partial W}{\partial t} + \sum_{i=1}^{k} M_i A_{i0} + H\left(t, q_1, \ldots, q_n, \frac{\partial W}{\partial q_1} + \sum_{i=1}^{k} M_i A_{i1}, \ldots, \frac{\partial W}{\partial q_n} + \sum_{i=1}^{k} M_i A_{in}\right) = 0 \tag{1.5}$$

where a_j and b_j are constants.

Strictly speaking, the relations (1.3) represent the integrals of a system of equations obtained from (1.2) by replacing in the latter λ_i by $M_i^{\cdot}$ $(i = 1, \ldots, k)$. Suslov has shown that along the trajectories of Eqs. (1.2) we have $M_i = \int \lambda_i dt$ $(i = 1, \ldots, k)$ and in the third chapter of [2] he gave another proof of these formulas, using however the fact that (1.3) represent the first integrals of (1.2).

The author of [4 - 10] obtained certain conditions of existence of the integrals of the type (1.3) and (1.4) of Eqs. (1.2) for the systems with nonholonomic homogeneous constraints (1.1) ($A_{i0} = 0$, $i = 1, \ldots, k$), first for the scleronomous system [6] and later for the rheonomic systems [9]. Seeking the integrals of (1.2) in the form

$$p_j = \pi_j(t, q_1, \ldots, q_n) \qquad (j = 1, \ldots, n) \tag{1.6}$$

he gave the name of "potential" [6] to the method of integrating the equations in which the relation

$$\sum_{j=1}^{n} \pi_j q_j^{\cdot} - H(t, q_1, \ldots, q_n, \pi_1, \ldots, \pi_n) = \frac{dW}{dt} \tag{1.7}$$

holds along the solution of (1.2).

The Hamilton-Jacobi method of integrating the equations of motion of an unrestricted system represents a particular example of the "potential" method and the conditions $\Omega_{jh} = \partial\pi_j / \partial q_h - \partial\pi_h / \partial q_j = 0$, $(j, h = 1, \ldots, n)$ which hold, indicate that the impulse field is irrotational.

We feel however that the name "potential method" is not justified, although some authors employ it [9, 14], since some of the quantities Ω_{jh} may differ from zero even in the case of a restricted holonomic system. If (1.7) holds, then [10, 12] by virtue of (1.1) the relations (1.3) hold together with

$$\sum_{i=1}^{k}\sum_{j=1}^{n} M_i A_{ij}\left(\delta q_j^{\cdot} - \frac{d}{dt}\delta q_j\right) = 0 \tag{1.8}$$

in which δq_j $(j = 1, \ldots, n)$ denote possible translations of the system.

For scleronomous systems with the constraint equations solved for $q_1^{\cdot}, \ldots, q_k^{\cdot}$,

$$\sum_{j=1}^{n} A_{ij} q_j^{\cdot} = 0 \quad (A_{ij} = \delta_{ij} \quad \text{for } j \leqslant k; \quad A_{ij} = -a_{ij}(q_1, \ldots, q_n) \quad \text{for } j > k) \tag{1.9}$$

$$(i = 1, \ldots, k < n - 1)$$

we can set

$$\delta q_s^{\cdot} - \frac{d}{dt}\delta q_s = 0 \qquad (s = k+1, \ldots, n)$$

$$\delta q_i^{\cdot} - \frac{d}{dt}\delta q_i = \sum_{(s,r)}^{k+1,\ldots,n} \Lambda^i_{(s,r)}(q_s^{\cdot}\delta q_r - q_r^{\cdot}\delta q_s) \quad (i = 1, \ldots, k)$$

where

$$\Lambda^i_{(s,r)} = \sum_{l=1}^{k} \frac{\partial a_{is}}{\partial q_l} a_{lr} + \frac{\partial a_{is}}{\partial q_r} - \sum_{l=1}^{k} \frac{\partial a_{ir}}{\partial q_l} a_{ls} - \frac{\partial a_{ir}}{\partial q_s} \tag{1.10}$$

As we know, the necessary and sufficient condition of integrability of (1.9) can be obtained by equating all $\Lambda^i_{(s,r)}$ to zero. Using this notation we can now write (1.8) in the form

$$\sum_{i=1}^{k}\sum_{(s,r)}^{k+1,\ldots,n} M_i \Lambda^i_{(s,r)}(q_s^{\cdot}\delta q_r - q_r^{\cdot}\delta q_s) = 0 \tag{1.11}$$

Hence, in accordance with an assertion in [10, 12] by virtue of independence of the quantities $\omega_{s,r} = q_s^{\cdot}\delta q_r - q_r^{\cdot}\delta q_s$ $(k+1 \leqslant s < r = k+2, \ldots, n)$ it follows that

$$\sum_{i=1}^{k} M_i \Lambda^i_{(s,r)} = 0 \qquad (k+1 \leqslant s < r = k+2, \ldots, n) \tag{1.12}$$

Let us set $n - k > 2$ and consider the above derivation in more detail. Conditions (1.12) are obtained for each pair of indices σ_1 and $\sigma_2 (k+1 \leqslant \sigma_1 < \sigma_2 = k+2, \ldots, n)$ from (1.11) with $\omega_{\sigma_1,\sigma_2} \neq 0$ and $\omega_{s,r} = 0$ $(k+1 \leqslant s < r = k+2, \ldots, n;$ $s \neq \sigma_1, r \neq \sigma_2)$, consequently they hold when $q^{\cdot}_{\sigma_3} = 0$ $(\sigma_3 = k+1, \ldots, n; \sigma_3 \neq \sigma_1, \sigma_3 \neq \sigma_2)$ since $q^{\cdot}_{\sigma_3}\omega_{\sigma_1,\sigma_2} + q^{\cdot}_{\sigma_2}\omega_{\sigma_3,\sigma_1} + q^{\cdot}_{\sigma_1}\omega_{\sigma_2,\sigma_3} \equiv 0$. On the other hand the functions M_i $(i = 1, \ldots, k)$ which can be found using the procedure described above are linear forms of $\partial W / \partial q_j$ $(j = 1, \ldots, n)$ but, according to (1.3), $\partial W / \partial q_j$ depend on the impulses and hence on the velocities $q_h^{\cdot}$ $(h = 1, \ldots, n)$. Therefore the conditions (1.12) are not necessary when $n - k > 2$.

Since the quantities δq_r $(r = k+1, \ldots, n)$ are independent, the following necessary conditions yield from (1.11):

$$\sum_{i=1}^{k}\sum_{s=k+1}^{n} M_i\Lambda^i_{(s,r)}q_s^{\cdot}=0 \qquad (r=k+1,\ldots,n)$$

which on the basis of the relations (1.2) and (1.3) can be represented in the form

$$\sum_{i=1}^{k}\sum_{s=k+1}^{n} M_i\Lambda^i_{(s,r)}\frac{\partial H}{\partial p_s}=0 \text{ when } p_j=\frac{\partial W}{\partial q_j}+\sum_{l=1}^{k} M_lA_{lj} \ (j=1,\ldots,n) \tag{1.13}$$
$$(r=k+1,\ldots,n)$$

It was noted in [13] that the conditions of existence of the integrals (1.3) in particular the conditions (1.12)) given in [6, 9, 10, 12] are regarded by the authors of [10, 11] as both,necessary and sufficient, without any reservations. The author of [13] used the equations of motion (1.2) as the basis for assuming that,on one hand $M_l^{\cdot}=\lambda_l$ $(l=1,\ldots,\ldots,k)$, and on the other hand M_l can be determined by means of the procedure given above. The relations (1.3) were differentiated with respect to the time and concluded that the necessary and sufficient conditions could be reduced to a single condition of integrability of the constraints (1.1) (see also [17]). This argument is however false. The author of [11] gave an example of a nonholonomic system in which the relations of the form (1.3) and (1.4) are integrals of motion.

Naziev (see abstract) similarly employed both of the above definitions of the functions M_l without however proving their equivalence. We note e.g. that for the holonomic systems with constraints (1.9) (below we show that the integrals (1.3) and (1.4) define the general solution of the equations of motion of such systems), $M_l^{\cdot}\neq\lambda_l$ for at least one combination of the indices (see 2.8)) when $\partial a_{is}/\partial q_l\neq 0$ $(i,\ l=1\ \ldots,k;\ s=k+1,\ \ldots,n)$ Therefore Naziev's necessary and sufficient conditions of existence of the integrals of the form (1.3) and (1.4)

$$\sum_{i=1}^{k} M_i\left(\frac{\partial A_{i\alpha}}{\partial q_\beta}-\frac{\partial A_{i\beta}}{\partial q_\alpha}\right)=0 \qquad (\alpha,\beta=0,1,\ldots,n), \qquad q_0=t) \tag{1.14}$$

are not necessary in the general case (1.1). For the stationary, Chaplygin-type constraints (1.9) $a_{ij}=a_{ij}(q_{k+1},\ldots,q_n)$ $(i=1,\ldots,k;\ j=1,\ldots,n)$ the formulas (1.12) and (1.14) coincide.

Thus the question of sufficiency of conditions (1.12) remains open.

2. For a scleronomous system whose kinetic energy is a homogeneous form of the velocities, the equations of constraints (1.9) in canonical variables have the form

$$\sum_{j,\ h=1}^{n} A_{ij}\frac{\partial^2 H}{\partial p_h\partial p_j}p_h=0 \qquad (i=1,\ldots,k) \tag{2.1}$$

Let us consider the quantities

$$\varepsilon_h=p\quad -\frac{\partial W}{\partial q_h}-\sum_{l=1}^{k} M_lA_{lh} \qquad (h=1,\ldots,n)$$

in which the functions M_i represent a solution of the following system of equations

$$\sum_{j,\ h=1}^{n} A_{ij}\frac{\partial^2 H}{\partial p_h\partial p_j}\left(\frac{\partial W}{\partial q_h}+\sum_{l=1}^{k} M_lA_{lh}\right)=0 \quad (i=1,\ldots,\ k) \tag{2.2}$$

and $W(t, q_1, \ldots, q_n, a_1, \ldots, a_n)$ is an integral of (1.5) satisfying the auxilliary system (1.13) For this reason some of the constants $a_1, \ldots, a_n$ are not arbitrary.

From (2.1) and (2.2) it follows that

$$\sum_{j,\, h=1}^{n} A_{ij} \frac{\partial^2 H}{\partial p_h \partial p_j} \varepsilon_h = 0 \qquad (i = 1, \ldots, h) \tag{2.3}$$

We find the derivatives $d\varepsilon_h / dt$ $(h = 1, \ldots, n)$ using Eqs. (1.2) and (1.9)

$$\varepsilon_h^{\cdot} = -\frac{\partial H}{\partial q_h} + \sum_{l=1}^{k} \lambda_l A_{lh} - \frac{\partial^2 W}{\partial t \partial q_h} - \sum_{j=1}^{n} \frac{\partial^2 W}{\partial q_j \partial q_h} \frac{\partial H}{\partial p_j} - \tag{2.4}$$

$$-\sum_{l=1}^{k} M_l^{\cdot} A_{lh} - \sum_{s=k+1}^{n} \sum_{l=1}^{k} M_l \left(\sum_{i=1}^{k} \frac{\partial A_{lh}}{\partial q_i} a_{is} + \frac{\partial A_{lh}}{\partial q_s} \right) \frac{\partial H}{\partial p_s} \qquad (h = 1, \ldots, n)$$

From (1.5) we have

$$\frac{\partial^2 W}{\partial q_h \partial t} + \frac{\partial H}{\partial q_h} + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \left(\frac{\partial^2 W}{\partial q_h \partial q_j} + \sum_{l=1}^{k} \frac{\partial M_l}{\partial q_h} A_{lj} + \sum_{l=1}^{k} M_l \frac{\partial A_{lj}}{\partial q_h} \right) = 0 \tag{2.5}$$

$$(h = 1, \ldots, n)$$

Comparing (2.4) with (2.5) and using (1.2), (1.9), (1.10) and (1.13) we obtain

$$\varepsilon_h^{\cdot} = \sum_{l=1}^{k} A_{lh} \mu_l + \sum_{j=1}^{n} \sum_{l=1}^{k} \sum_{r,s=k+1}^{n} \delta_{hr} M_l \Lambda_{(r,s)}^{l} \frac{\partial^2 H}{\partial p_j \partial p_s} \varepsilon_j \qquad (h = 1, \ldots, n)$$

$$\mu_l = \lambda_l - M_l^{\cdot} - \sum_{i=1}^{k} \sum_{s=k+1}^{n} M_i \frac{\partial a_{is}}{\partial q_l} q_s^{\cdot} \tag{2.6}$$

Let us differentiate (2.3) with respect to time. By (1.2) and (2.6) we have

$$\sum_{l=1}^{k} \sum_{j,h=1}^{n} A_{ij} \frac{\partial^2 H}{\partial p_h \partial p_j} A_{lh} \mu_l = \Phi_i (q_1, \ldots, q_n, p_1, \ldots, p_n, \varepsilon_1, \ldots, \varepsilon_n) \tag{2.7}$$

where the functions Φ_i represent certain forms linear and homogeneous in ε_h. Consequently, using (2.7) to eliminate μ_i $(i = 1, \ldots, k)$ from (2.6) we obtain a system of equations homogeneous in $\varepsilon_h(t)$ $(h = 1, \ldots, n)$. When $\varepsilon_h(t_0) = 0$ $(h = 1, \ldots, n)$, this system has a unique solution $\varepsilon_h(t) \equiv 0$, i.e. relations (1.3) represent the first integrals of (1.2). But then from (2.7) it follows that $\mu_l(t) \equiv 0$ $(l = 1, \ldots, k)$ or

$$\lambda_l = M_l^{\cdot} + \sum_{i=1}^{k} \sum_{s=k+1}^{n} M_i \frac{\partial a_{is}}{\partial q_l} q_s^{\cdot} \qquad (l = 1, \ldots, k) \tag{2.8}$$

Let the integral W contain $q \leqslant n$ arbitrary constants a_γ. We shall show that

$$\partial W / \partial a_\gamma = b_\gamma \qquad (\gamma = 1, \ldots g, \quad b_\gamma = \text{const}) \tag{2.9}$$

also represent the first integrals of (1.2). Let us insert W into (1.5) and differentiate the result with respect to a_γ

$$\frac{\partial^2 W}{\partial a_\gamma \partial t} + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \left(\frac{\partial^2 W}{\partial a_\gamma \partial q_j} + \sum_{l=1}^{k} \frac{\partial M_l}{\partial a_\gamma} A_{lj} \right) = 0 \qquad (\gamma = 1, \ldots, q)$$

Hence on the basis of (1.2) and (1.9) we obtain

$$\frac{\partial^2 W}{\partial a_\gamma \partial t}+\sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \frac{\partial^2 W}{\partial a_\gamma \partial q_j}=0 \qquad (\gamma=1,\ldots,q)$$

The above identities mean that the total time derivatives of (2.9) are, by virtue of (1.2), equal to zero.

Since the conditions (1.13) hold when the relations (1.12) exist, the latter represent the sufficient condition for (1.3) and (2.9) to be the first integrals of (1.2).

3. If a certain function W satisfies

$$f\left(q_1,\ldots,q_n,W,\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}\right)=0, \qquad F\left(q_1\ldots,q_n,W,\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}\right)=0$$

it also satisfies

$$[F,f]=\sum_{j=1}^{n}\left(\frac{\partial F}{\partial\left(\frac{\partial W}{\partial q_j}\right)}\left(\frac{\partial f}{\partial q_j}+\frac{\partial W}{\partial q_j}\frac{\partial f}{\partial W}\right)-\frac{\partial f}{\partial\left(\frac{\partial W}{\partial q_j}\right)}\left(\frac{\partial F}{\partial q_j}+\frac{\partial W}{\partial q_j}\frac{\partial F}{\partial W}\right)\right)=0$$

Here $[F, f]$ is the Jacobi bracket which becomes the Poisson bracket when $\partial f / \partial W = \partial F / \partial W = 0$ The proof of this theorem is based on the fact [15] that substitution of the solution W into the equations $f=0$ and $F=0$ converts the latter into identities.

Relations (1.13) obtained from Eq. (1.7), equations of constraints (1.9) and the D'Alembert principle are not, generally speaking, identities as $q_1,\ldots,q_n$ entering these relations are not arbitrary but represent a solution of (1.2). For this reason the Poisson brackets composed of the left hand sides of Eqs. (1.13) need not necessarily be equal to zero. The Poisson brackets for (1.12) are not zero for the same reason.

Example. Let us consider the problem of a homogeneous sphere rolling on a horizontal plane [16]. We assume the sphere to be of unit mass and denote its radius by a. The position of the sphere is defined by two Cartesian coordinates $q_1=x$ and $q_2=y$ of the center of the sphere relative to a fixed coordinate system, the xy-plane of which is horizontal, and by three Euler angles $q_3=\varphi$, $q_4=\psi$ and $q_5=\theta$. The kinetic energy of the sphere and the equations of constraints expressing the fact that the velocity of the point of contact of the sphere with the plane is zero, are

$$T={}^1\!/_2\,[q_1^{\cdot 2}+q_2^{\cdot 2}+{}^2\!/_5 a^2(q_3^{\cdot 2}+q_4^{\cdot 2}+q_5^{\cdot 2}+2q_3^{\cdot}q_4^{\cdot}\cos q_5)]$$
$$q_1^{\cdot}+a(q_3^{\cdot}\sin q_5\cos q_4-q_5^{\cdot}\sin q_4)=0$$
$$q_2^{\cdot}+a(q_3^{\cdot}\sin q_5\sin q_4+q_5^{\cdot}\cos q_4)=0$$

If the forces applied to the system are not potential, the equation of motion can be written as

$$q_j^{\cdot}=\frac{\partial T}{\partial p_j}, \quad p_j^{\cdot}=-\frac{\partial T}{\partial q_j}+Q_j+\sum_{i=1}^{k}\lambda_i A_{ij} \qquad (j=1,\ldots,n) \tag{3.1}$$

where Q_j denote the generalized forces. When $Q_j=\partial U / \partial q_j$ $(j=1,\ldots,n)$, Eqs.(3.1) coincide with (1.2) $(H=T-U)$.

Having differentiated the equations of constraints with respect to time we can use (3.1) to determine the values of the multipliers and, consequently, the constraint reactions

$$R_j=\sum_{i=1}^{k}\lambda_i A_{ij} \qquad (j=1,\ldots,n)$$

which in the present example are given in the form

$$R_1 = \frac{5}{7a}(Q_4 \operatorname{ctg} q_5 - Q_3 \operatorname{cosec} q_5)\cos q_4 + \frac{5}{7a} Q_5 \sin q_4 - \frac{2}{7} Q_1$$

$$R_2 = \frac{5}{7a}(Q_4 \operatorname{ctg} q_5 - Q_3 \operatorname{cosec} q_5)\sin q_4 - \frac{5}{7a} Q_5 \cos q_4 - \frac{2}{7} Q_2$$

$$R_3 = \frac{5}{7}(Q_4 \cos q_5 - Q_3) - \frac{2a}{7}(Q_1 \cos q_4 + Q_2 \sin q_4)\sin q_5$$

$$R_4 = 0$$

$$R_5 = -\frac{5}{7} Q_5 + \frac{2a}{7}(Q_1 \sin q_4 - Q_2 \cos q_4) \tag{3.2}$$

while the conditions (1.12) become [10]

$$\cos q_4 \frac{\partial W}{\partial q_1} + \sin q_4 \frac{\partial W}{\partial q_2} + \frac{5}{2a \sin q_5}\frac{\partial W}{\partial q_3} - \frac{5 \cos q_5}{2a \sin q_5}\frac{\partial W}{\partial q_4} = 0$$

$$\sin q_4 \frac{\partial W}{\partial q_1} - \cos q_4 \frac{\partial W}{\partial q_2} - \frac{5}{2a}\frac{\partial W}{\partial q_5} = 0 \tag{3.3}$$

From (2.2) it follows that the force function U does not influence the dependence of M_i $(i = 1, \ldots, k)$ on $\partial W/\partial q_j$ $(j = 1, \ldots, n)$.

The following statement concerning the system (3.3) was made in [10]: "By constructing the Poisson bracket it can easily be shown that the resulting system is incompatible (by this the authors mean that $W =$ const). Consequently, the potential method of integration is not applicable to the nonholonomic systems in question." This is however false in the case of an inertial motion of a sphere, i.e. when $Q_j = 0$ $(j = 1, \ldots, 5)$. Indeed, in this case the reactions (3.2) are equal to zero, equations (1.2) assume the Hamiltonian form $(H = T)$, the relations (1.3) - (1.5) hold and the conditions (3.3) in which $M_1 = M_2 = 0$ also hold.

The following is also of interest. Let Q_j $(j = 1, \ldots, 5)$ be independent of velocities, then the reactions (3.2) are also independent of velocities. Clearly, Eqs. (3.1) of motion of a sphere are Hamiltonian only when the resultant force $Q_j + R_j$ $(j = 1, \ldots, 5)$ is potential. We consider two cases

$$1)\quad Q_1 = Q_2 = 0,\quad Q_3 = \frac{\partial U}{\partial q_3},\quad Q_4 = \frac{\partial U}{\partial q_4},\quad Q_5 = \frac{\partial U}{\partial q_5},\quad U = U(q_3, q_4, q_5)$$

If (3.1) are Hamiltonian, then we have the following particular relation

$$\frac{\partial R_1}{\partial q_4} = \frac{\partial R_2}{\partial q_4} = \frac{\partial R_3}{\partial q_4} = \frac{\partial R_5}{\partial q_4} = 0$$

from which it follows that $Q_4 \cos q_5 - Q_3 = 0$ and $Q_5 = 0$, which implies that $U = \text{const}$.

$$2)\quad Q_1 = Q_1(q_1, q_2)\quad Q_2 = Q_2(q_1, q_2),\quad Q_3 = Q_4 = Q_5 = 0$$

If (3.1) are Hamiltonian, then we have the following particular relation

$$\partial R_5 / \partial q_4 = \partial^2 R_5 / \partial q_4^2 = 0$$

from which $Q_1 = Q_2 = 0$ follows.

Thus, when the dynamic forces applied to the sphere can be reduced to a potential moment or to a resultant force passing through the center of the sphere, the system (3.1) is not Hamiltonian.

4. Let us assume that conditions (1.13) hold. Then Eqs. (1.2) can be written in the Hamiltonian form. Indeed, using (1.2), (1.3), (1.13) and (2.8) we obtain

$$\sum_{i=1}^{k} \lambda_i A_{ij} = \frac{d}{dt}\frac{\partial L_0}{\partial q_j^{\bullet}} - \frac{\partial L_0}{\partial q_j} \qquad (j = 1, \ldots, n), \qquad L_0 = \sum_{h=1}^{n}\sum_{l=1}^{k} M_l A_{lh} q_h^{\bullet}$$

Therefore the equations of motion of the system written in the form of Lagrange equations with constraint multipliers assume the form of the Lagrange equations of the second kind with the function $L_1 = L - L_0$. These in turn can be transformed into the Hamilton equations equivalent to (1.2), by introducing the generalized impulses

$$P_j = \frac{\partial L_1}{\partial q_j^{\bullet}} = p_j - \sum_{i=1}^{k} M_i A_{ij} \qquad (j = 1, \ldots, n)$$

The Hamiltonian is

$$H_1 = \sum_{j=1}^{n} P_j q_j^{\bullet} - L_1 = \sum_{j=1}^{n} p_j q_j^{\bullet} - L =$$

$$= H\left(q_1, \ldots, q_n, \quad P_1 + \sum_{i=1}^{k} M_i A_{i1}, \ldots, P_n + \sum_{i=1}^{k} M_i A_{in}\right)$$

If the complete integral W_1 of the Hamilton-Jacobi equation

$$\frac{\partial W_1}{\partial t} + H\left(q_1, \ldots, q_n, \frac{\partial W_1}{\partial q_1} + \sum_{i=1}^{k} M_i A_{i1}, \ldots, \frac{\partial W_1}{\partial q_n} + \sum_{i=1}^{k} M_i A_{in}\right) = 0$$

has been found, then $P_j = \partial W_1 / \partial q_j$ $(j = 1, \ldots, n)$. However, comparing these expressions with (1.3) we find that $\partial W_1 / \partial q_j = \partial W / \partial q_j$ $(j = 1, \ldots, n)$, therefore $P_j = \partial W / \partial q_j$; the Hamilton-Jacobi equation coincides with (1.5) $(A_{i0} = 0,\ i = 1, \ldots, k)$.

Conditions (1.13) are obviously the first integrals of the following equations of motion

$$q_j^{\bullet} = \frac{\partial H}{\partial P_j}, \qquad P_j^{\bullet} = -\frac{\partial H}{\partial q_j} \qquad (j = 1, \ldots, n)$$

For this reason, by the Poisson theorem, the Poisson brackets constructed for the left-hand parts of (1.13) remain constant along the trajectories of the system. In the previous section we have shown that these constants must not be assumed simultaneously equal to zero.

5. The compatibility of (1.5) and (1.13) can be investigated as follows. Let W be the complete integral of (1.5). From (1.4) we find

$$q_j = \varphi_j(t, a_1, \ldots, a_n, \quad b_1, \ldots, b_n) \qquad (j = 1, \ldots, n) \tag{5.1}$$

choosing the constants $a_1, \ldots, a_n, b_1, \ldots, b_n$ so as to satisfy the equations of constraints at $t = t_0$. The other $2n - k$ constants remain arbitrary. We now insert Eqs. (5.1) into (1.13). If the resulting relations are satisfied identically, then in accordance with Sect. 2 the formulas (5.1) represent the general solution of (1.2). If, however, the relations obtained impose some restrictions on the remaining $2n - k$ arbitrary constants and $W \neq$ const, then according to Sect. 2 the formulas (5.1) represent a particular solution of (1.2).

6. The proposed method of integrating the equations of motion with constraint multipliers represents a generalization of the Hamilton-Jacobi method, and becomes identical to it when $M_i = 0$ $(i = 1, \ldots, k)$. In the latter case the method yields only such motions of the system, in which the constraint reactions (2.8) are zero. For systems with two degrees of freedom ($n - k = 2$) conditions (1.13) assume the form (1.12) (in some cases the conditions (1.13) and (1.12) remain equivalent even if $n - k > 2$). When $k \geqslant 2$, these conditions may hold when $M_i \neq 0$ $(i = 1, \ldots, k)$ [11].

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PLANE PERTURBED MOTION OF A MATERIAL POINT OF VARIABLE MASS

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L. N. GRUDTSYN
(Saratov)
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Motion of a material point of variable mass in a central force field in the presence of a perturbing force is considered. The solution of the problem is obtained in quadratures.

1. We consider a motion of a material point of variable mass in a central perturbing field obeying the following law

$$P(r) = \lambda m r^{-n} \tag{1.1}$$

where λ and n are the constant field characteristics (when $n = 2$ and $\lambda < 0$, we have the case of a Newtonian gravitational field), m is the mass of the material point and r is its distance from the center.

We assume that the mass of the point is a continuously differentiable function of its distance from the center

$$m = m_0 f(r), \qquad r = r(t) \qquad (m = m_0,\ r = r_0, \text{ for } t = 0) \tag{1.2}$$

and we also assume that

$$\mathbf{u} = p(r)\,\mathbf{v} \tag{1.3}$$

where $\mathbf{v}$ is the velocity of motion of the point in the inertial frame of reference, $\mathbf{u}$ is the velocity of the particles rejected (or assimilated) by the parent point up to the given instant and $p(r)$ is a specified continuous function. Then the reaction force can be written as

$$\mathbf{R}(r) = m_0 r^{\cdot} f'(r)\, g(r)\, \mathbf{v}, \qquad g(r) = p(r) - 1 \tag{1.4}$$

where a dot denotes derivative with respect to time and a prime, with respect to r.
We assume that in addition to the forces given, a perturbing force $\mathbf{F}^*$ lying in the plane of the trajectory and orthogonal to the vector $\mathbf{v}$ is also acting on the point. An analogous problem of motion of a point of constant mass was studied in [1], where it was shown that if

$$\mathbf{F}^* = m_0 \mathbf{F}(r, v) \tag{1.5}$$

then the problem can be reduced to quadratures. We shall show that this remains true for the case of a point of variable mass.

Equations of plane motion of a point under the conditions (1.1) - (1.3) have the following form in the polar coordinates:

$$r^{\cdot\cdot} - r\varphi^{\cdot 2} = \frac{\lambda}{r^n} + \frac{1}{f} F_r + \frac{f'}{f} g r^{\cdot 2}, \qquad (r^2\varphi^{\cdot})^{\cdot} = \frac{r}{f} F_\varphi + \frac{f'}{f} g r^2 r^{\cdot} \varphi^{\cdot} \tag{1.6}$$

$$F_r = -F(r, v)\frac{r\varphi^{\cdot}}{v}, \qquad F_\varphi = F(r, v)\frac{r^{\cdot}}{v}, \qquad v^2 = r^{\cdot 2} + r^2\varphi^{\cdot 2}$$